Can it be done?

Miracle Sudoku on the Torus

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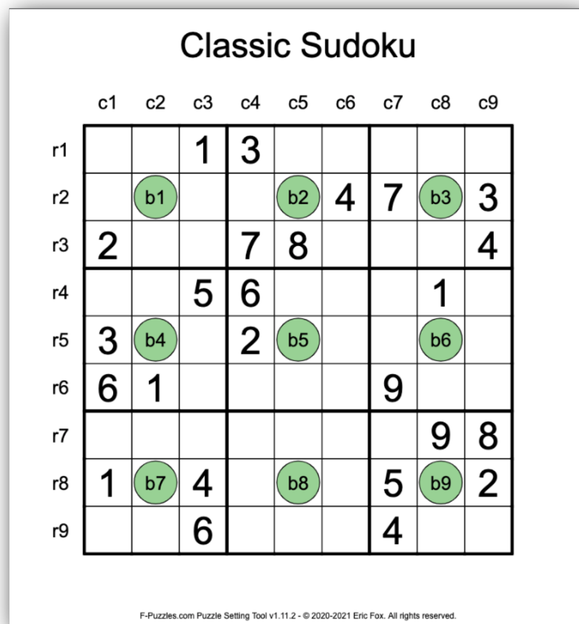

You're probably familiar with Sudoku. Sudoku is a puzzle game played on a 9x9 grid. The grid is broken up into a 3x3 set of 3x3 boxes. There are some numbers given in the grid when you start. Your goal is to fill in the rest of the grid while ensuring that each row contains the numbers 1 through 9, each column contains the numbers 1 through 9, and each box contains the numbers 1 through 9.

The grid at the right[1] shows a sample puzzle with the rows, columns, and boxes labelled. Most puzzles will not have those labels (or the green circles). Those are not part of the puzzle, but will help in describing things below.

The description above is ordinary, Classic Sudoku. There are lots and lots of variants of Sudoku out there which add to or modify the above rules in some way.

Miracle Sudoku

One very interesting variant is called Miracle Sudoku. This variant was created by Mitchell Lee and was featured on the Cracking The Cryptic[2] YouTube channel in May of 2020.

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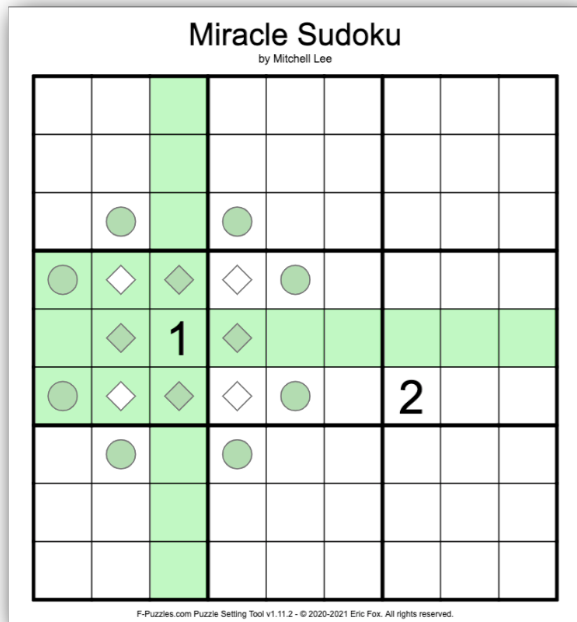
[1] Sudoku diagrams created using f-puzzles.com v1.11.2 ©2020-2021 by Eric Fox

[2] https://www.youtube.com/watch?v=yKf9aUIxdb4

In Miracle Sudoku, the normal, classic rules of Sudoku apply. Additionally, if a digit appears in a cell, it cannot appear in any cell that it could reach if it were a chess king or a chess knight. Furthermore, orthogonally adjacent cells cannot contain consecutive digits.



Mitchell's puzzle is shown on the right with some additional notation to demonstrate the rules.

The cell in row 5, column 3 contains a 1. Because it contains a 1, none of the other shaded cells can contain a 1 by the rules of Classic Sudoku. (If there were a second 1 in one of those shaded cells, then there would not be enough remaining shaded cells for the digits 2 through 9 in the corresponding row, column, or box.)

Because this is Miracle Sudoku, there also cannot be a 1 in any of the cells marked with a diamond because it could get to those cells if it were a chess king. Also, because this is Miracle Sudoku, there cannot be a 1 in any of the cells marked with a circle as the 1 could get to those cells if it were a chess knight. Furthermore, in Miracle Sudoku, the cells marked with green diamonds cannot contain a digit that is consecutive with 1 (i.e. cannot contain the digit 2).

Because of these extra restrictions, the depicted puzzle has a unique solution even though there are only two given digits in the puzzle. In contrast, under the classic rules of Sudoku, a puzzle must have at least 17 given digits to have a unique solution.[3]

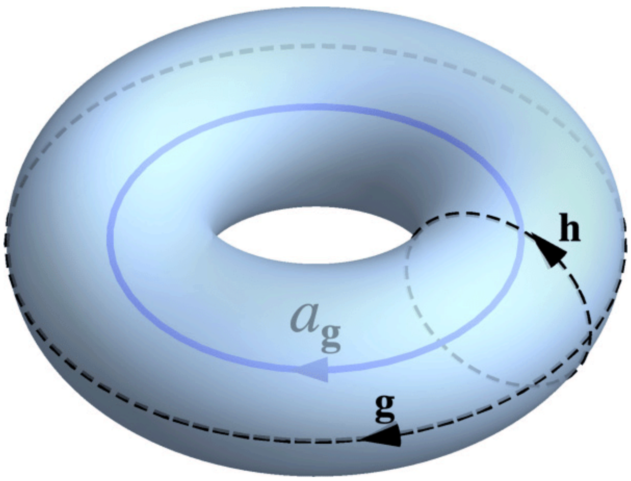
Side note on terminology

In Sudoku parlance, we say that the 1 "sees" all of those marked squares. By the rules of Sudoku and the rules of Miracle Sudoku, a digit is not allowed to see another instance of the same digit. (Miracle Sudoku also forbids a digit from sharing a cell's edge with a consecutive digit.)

[3] https://www.technologyreview.com/2012/01/06/188520/mathematicians-solve-minimum-Sudoku-problem/

Flat Torus

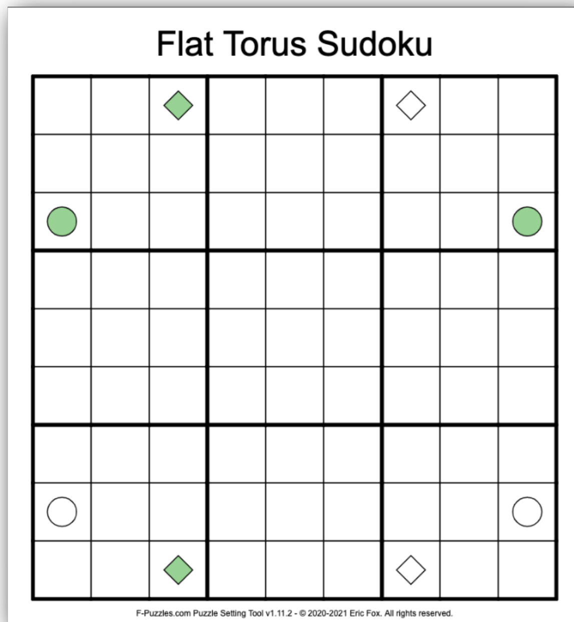
A torus is a donut shape. You are probably used to seeing the torus represented as a three-dimensional object. The surface of a torus, however, is only two-dimensional.



The image at the left[4] shows a torus. If it were made out of a flexible enough material, you could cut it along both dotted lines and flatten it out into a rectangle.

If you played video games at all in the ‘80s or ‘90s, you have an intuitive sense of this structure. In the flattened rectangle, points along the top edge of the rectangle had been, before the cut, adjacent to the point directly below them along the bottom edge. Likewise, points along the left edge of the rectangle had been adjacent to the point directly across from them on the right edge. When your space-ship went off the top of the screen, it came back on the bottom.

[4] Symmetry fractionalization, defects, and gauging of topological phases - Scientific Figure on ResearchGate. Available from: https://www.researchgate.net/figure/The-g-h-sector-on-a-torus-where-a-closed-g-defect-branch-line-wraps-around-the_fig7_335951735 [accessed 23 Apr, 2022]



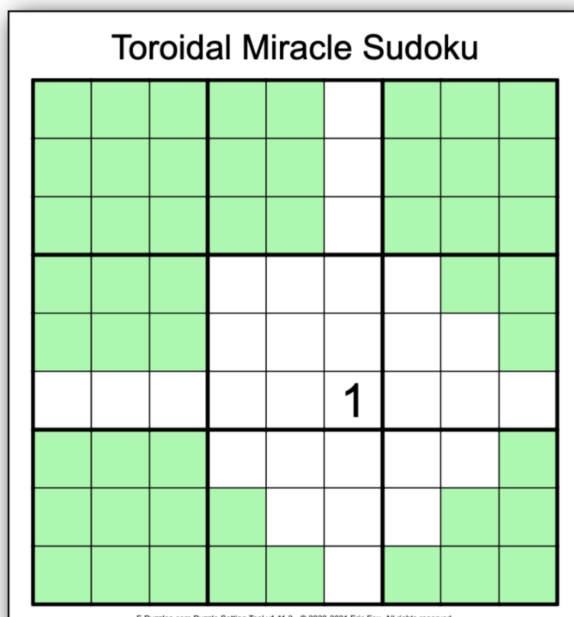
Usually, we think of playing Sudoku on a square. If we were playing on a flat torus, the top cell in a column would be adjacent to the bottom cell of the same column and the left-most cell in a row would be adjacent to the right-most cell in the same row.

In the example at the right, the shaded diamonds are adjacent as are the unshaded diamonds, the shaded circles, and the unshaded circles.

In Classic Sudoku, this doesn't change anything. The shaded diamond can already see the other shaded diamond by virtue of being in the same column.

In Miracle Sudoku, however, the shaded diamond at the top of column three can also see the bottom cells of columns two and four by chess king moves and the bottom cells of columns one and five by chess knight moves and the cells in row eight in columns two and four by chess knight moves, too.

Miracle Sudoku on the Flat Torus



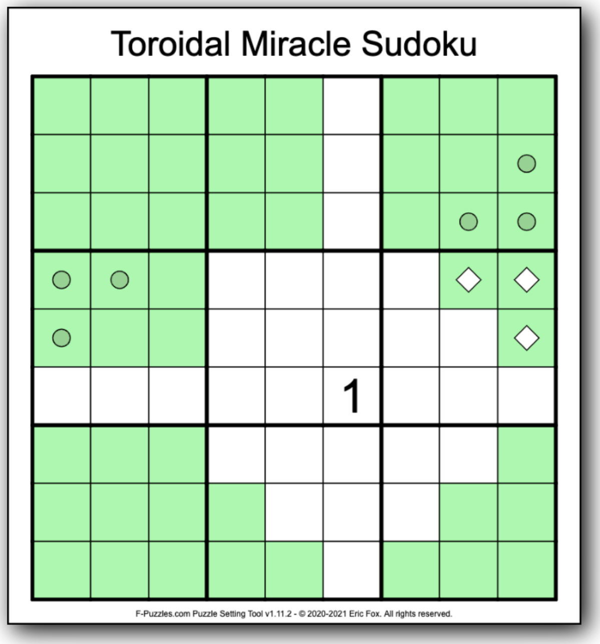
As shown in the previous section, some cells can see more other cells on a flat torus than they can on a square. On a flat square, the cell at the top left corner, for example, can only see 20 other cells. All of its chess king moves and chess knight moves overlap with the box it is in, so it sees the eight other cells in its box, the six cells in its row outside of its box, and the six cells in its column outside of its box. That same cell on the flat torus can see nine additional cells (three by chess king moves and six by chess knight moves).

Does that extra adjacency restrict us too much to form a puzzle? Let's find out.

In the final solution, there has to be a digit in every cell. Let's consider the cell at row six, column six. It has to contain a digit. For now, let's assume this digit is a 1. None of the following argument depends on which digit we choose here as we will not need Miracle Sudoku's restriction on consecutive digits.

That 1 can see all of the unshaded squares in the previous picture. This means that any additional 1's in this puzzle have to be in one of the shaded squares.

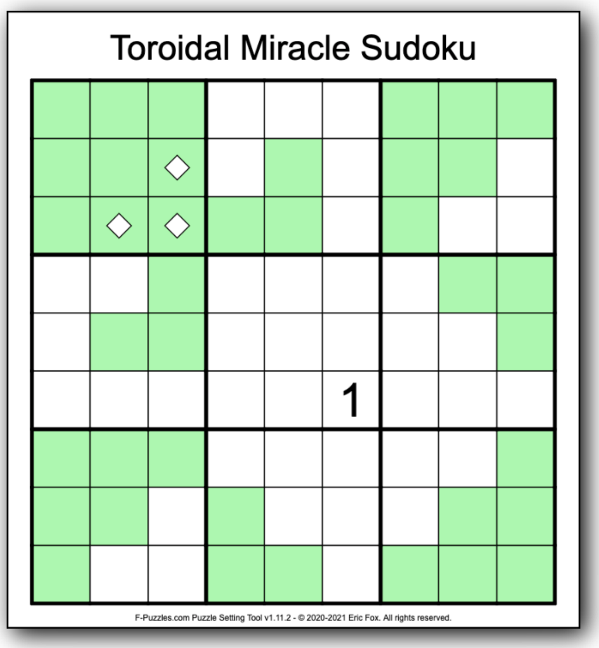
Let's consider the places a 1 could go in box six. These are marked with a diamond in the picture at the right.



Each of those cells can see the cells marked with a circle in the box above it by either Classic Sudoku rules, a chess king move, or a chess knight move. Because this is a flat torus, the same thing can happen wrapping around to box 4. So, all of the cells marked with a diamond can see all of the cells marked with a circle.

We know that there has to be a 1 in one of the cells marked with a diamond. As such, there **cannot** be a 1 in any of the cells marked with a shaded circle.

This reduces the places a 1 can go in the grid to only the shaded cells in the picture below.

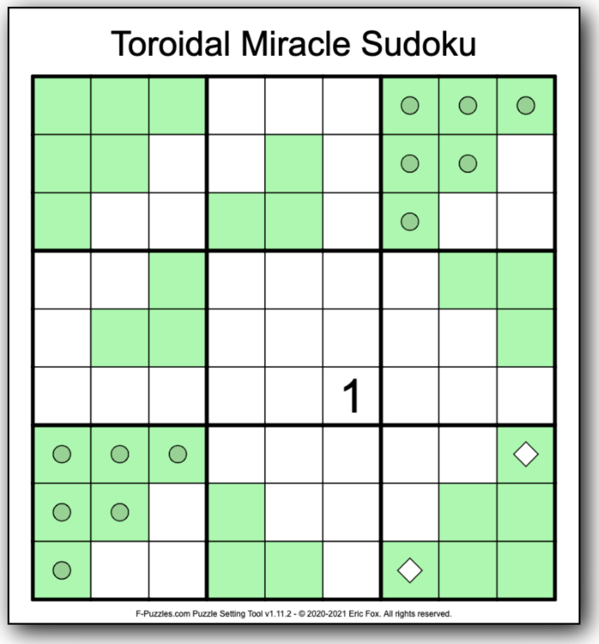


Now, let's consider the cells marked with a diamond in box 1 in the picture at the left. The two cells marked with a diamond in column 3 can see all of the green cells in box 2. So, if either of those were a 1, then there would be no place to put a 1 in box 2.

By a similar argument, the two cells marked with a diamond in row 3 can see all of the green cells in box 4. So, if either of those cells were a 1, then there would be no place to put a 1 in box 4.

This all means that we cannot put a 1 in any of the cells marked with a diamond in the picture on the left on the previous page.

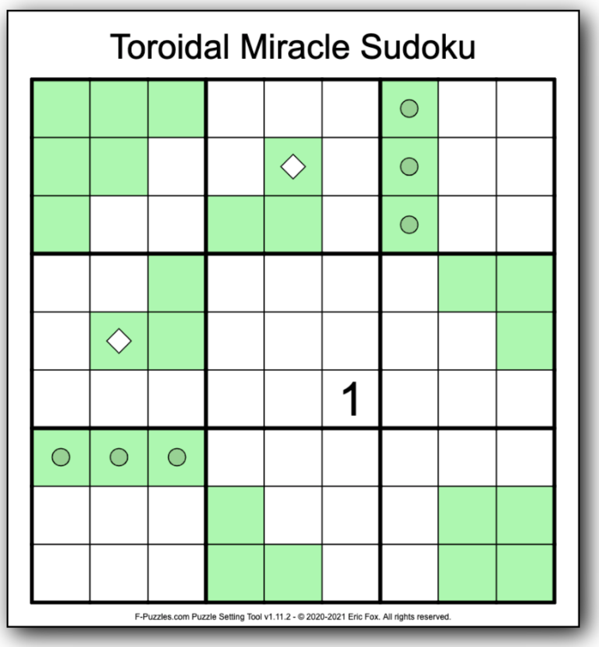
Next, let’s look at cell in row 9, column 7 that is marked with a diamond in the picture at the right.



By either Sudoku or chess king or chess knight moves, it can see all of the green squares in box 3. If we were to put a 1 in row 9, column 7, there would be no place left to put a 1 in box 3.

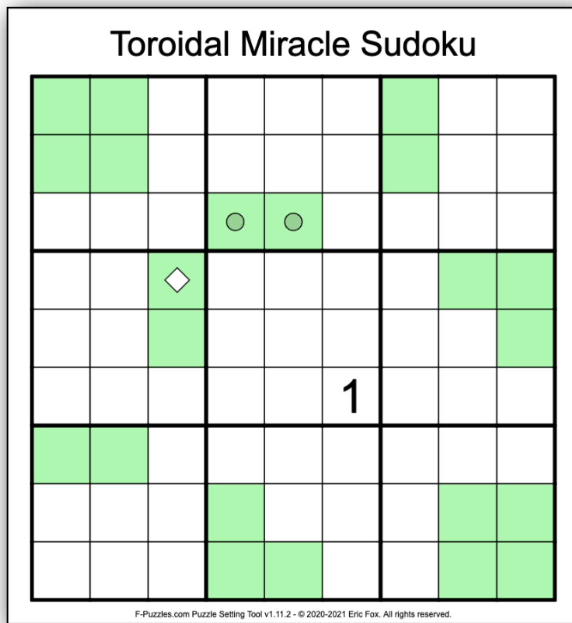
By a symmetric argument, there cannot be a 1 in row 7, column 9 without eliminating all of the places one could still place a 1 in box 7.

So, now we know we cannot put a 1 either the cell at row 7, column 9 or in the cell at row 9, column 7. Once we eliminate those, then we see that the only cells in column 7 which could contain a 1 are in box 3. That means that the 1 in box 3 has to be in column 7 (row 1, row 2, or row 3). A symmetric argument works horizontally on row 7. We have now concluded that any remaining 1’s in the puzzle must be confined to the cells still shaded green in the picture at the left.



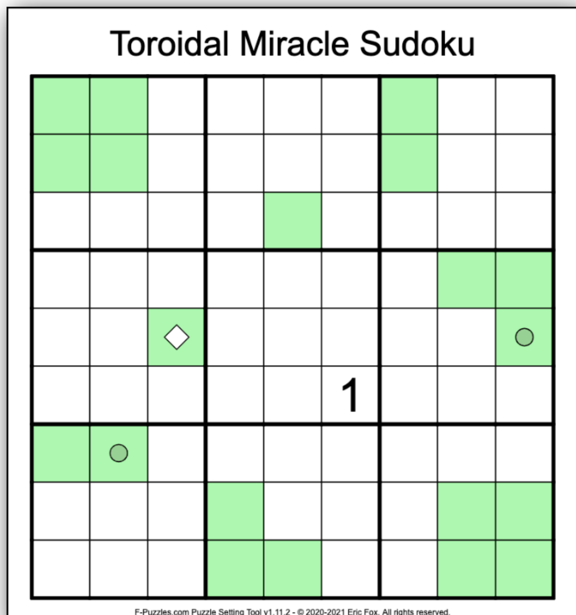
Now, we will consider the cell in row 2, column 5 which is marked with a diamond in the picture at the left. It can see all of the highlighted cells in box 3 either by Classic Sudoku or by a chess knight move. The symmetric argument applies to the cell at row 5, column 2. This means that neither of those cells marked with a diamond can be a 1.

Now, looking at the cell in row 4, column 3 which is marked with a diamond in the picture at the right. That cell can see both of the shaded cells in box 2. As such, if it were a 1, then there would be no place left to place a 1 in box 2.

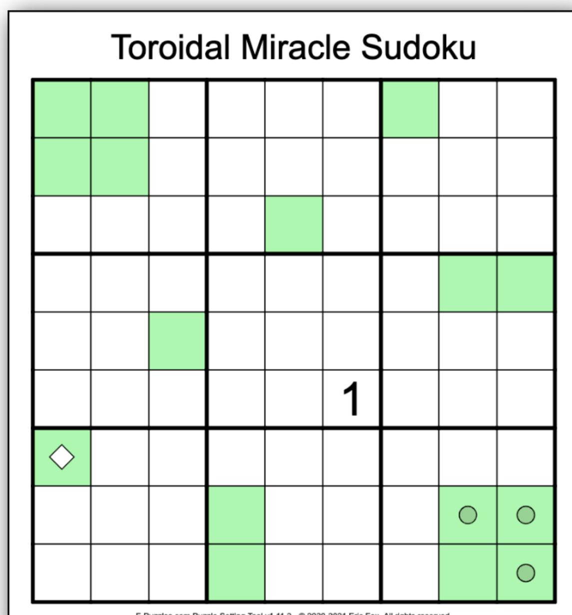


By a symmetric argument, there cannot be a 1 in row 3, column 4 either.

So, now we are in the situation in the image below.



The cell in row 5, column 3 which is marked with a diamond can be seen by the cell in row 5, column 9 by Classic Sudoku and the cell in row 7, column 2 by a chess knight move. So, neither of those cells can be a 1. The symmetric argument on row 3, column 5 means that neither row 2, column 7 nor row 9, column 5 can be a 1 either.



From there, we are left with only one place a 1 can go in box 7: row 7, column 1. That cell can see all of the cells in box 9 which are shaded with a circle either by a chess king move or a chess knight move. None of those cells marked with a circle can be a 1. The symmetric argument with the cell at row 1, column 7 eliminates the last highlighted cell in box nine.

Conclusion for the Torus

We have now shown that whatever digit appears in row 6, column 6 cannot appear in box 9.

Symmetric arguments show that no digit can appear in any corner of any box without forbidding that digit in another other box (the box with which that cell shares a corner but no edge).

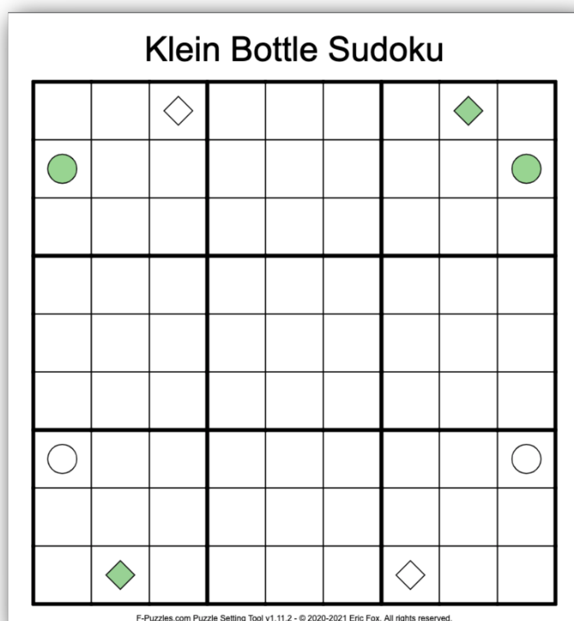
We leave it as an exercise to the reader to demonstrate that placing a digit on the edge of a box (but not the corner) is also impossible in Miracle Sudoku on the torus.

Once you have that, it is easy to see that you cannot even place a digit in the center of a box. If you did, it would rule out the two other box centers in its column and the two other box centers in its row. In those cells, the digit would have to appear on either an edge or a corner of the box.

Miracle Sudoku is completely impossible on the torus.

What about the Klein Bottle?

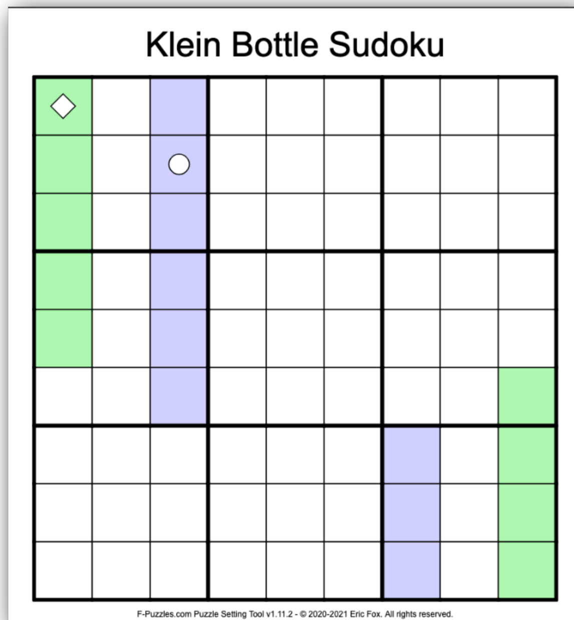
Now, you might be asking, what about a Klein bottle? A flat Klein bottle can be represented as a square where the left and right edges behave just like they did on the flat torus, but the top and bottom edges line up with their mirror images. The cell "above" row 1, column 1 is row 9, column 9. The cell "below" row 9, column 2 is row 1, column 8.



This causes a problem with the Classic Sudoku rules because now column 4 sees column 6. That means that if you put a digit in column 4, you cannot put that digit anywhere in column 6.

There are two ways we could deal with that problem. One way is to not allow the Classic Sudoku rules to wrap across the top and the bottom. This, however, makes cells on the edge of the grid behave differently than cells in the same spot in their box elsewhere on the grid. That's just not aesthetically pleasing.

A better way to deal with that problem is to say that a cell's column is that cell, the four cells "above" it and the four cells "below" it[5]. Thus, row 1, column 1 would be in the same "column" as rows 2, 3, 4, and 5 of column 1 and in the same "column" as rows 6, 7, 8, and 9 of column 9. It would not, however, be in the same column as rows 6, 7, 8, and 9 of column 1 (nor rows 2, 3, 4, and 5 of column 9). Two different cells are shown along with their "columns" in the picture to the right.



This has the odd effect that row 1, column 1 and row 2, column 1 are each in the other's "column", but they are not in the same "column". However, this gets us back to the situation we had on the flat torus where every cell's reach/visibility depends only on where it is in its box not on where its box is in the grid.

Tune in again next time (or, better yet, play with it on your own), to analyze classic and Miracle Sudoku with this new definition of "columns".

[5] I refuse to consider abominations like having a cell's "column" be the three cells "above" it, its own cell, and the five cells "below" it (or any other way you want to try to chiral columns).